Wreath products, nilpotent orbits and symplectic deformations

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Abstract

We recover the wreath product $X := \mathrm{Sym}^2(\mathbb{C}^2/\pm 1)$ as a transversal slice to a nilpotent orbit in $\mathfrak{sp}_6$. By using deformations of Springer resolutions, we construct a symplectic deformation of symplectic resolutions of X. AMS Classification: 14E15, 14M17

0. Introduction

Let $H \subset \mathrm{Sp}(2n)$ be a finite sub-group and $X := \mathbb{C}^{2n}/H$ the quotient symplectic variety. Given a projective symplectic resolution

$$Z \to X, \tag{1}$$

it was shown in [GK] that there exists a symplectic deformation of (1) over $B := H^2(Z, \mathbb{C})$, i.e. a morphism $\Pi : \mathcal{Z} \to \mathcal{X}$ over B such that over the origin $0 \in B$, $\Pi_0 : \mathcal{Z}_0 \to \mathcal{X}_0$ is the resolution (1), and over a generic point $b \in B$, $\mathcal{Z}_b, \mathcal{X}_b$ are symplectic smooth varieties isomorphic under Π_b, where Π_b is the restriction of Π to the fibers over b. The proof of this theorem is based on the infinitesimal and formal deformations of π developed in [KV] and the globalization is obtained by using the expanding $\mathbb{C}^*$-action on X. As noted already in [GK], this deformation is very similar to the deformation of the Springer resolution of nilpotent cones given by Grothendieck's simultaneous resolution ([Slo]). However, the construction of symplectic deformations in general is rather implicit. The purpose of this note is to provide some explicit examples of such deformations.

A class of important examples of symplectic resolutions is given by Hilbert-Chow morphisms ([Wan]): $\mathrm{Hilb}^n(\mathbb{C}^2//\Gamma) \to \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$, where $\Gamma \subset SL(2)$

is a finite sub-group and $\mathbb{C}^2//\Gamma \to \mathbb{C}^2/\Gamma$ is the minimal resolution. The simplest case is $n = 1$. It can be shown ([Slo]) that a transverse slice of the sub-regular nilpotent orbit in the nilpotent cone has ADE singularities, then Grothendieck's simultaneous resolution provides symplectic deformations of the minimal resolution (see also [GK] section 3).

The next simple case is $n = 2$ and $\Gamma = \pm 1$, i.e. the resolution $\pi : \mathrm{Hilb}^2(T^*\mathbb{P}^1) \to \mathrm{Sym}^2(\mathbb{C}^2/\pm 1)$. Our aim of this note is to construct a symplectic deformation of the resolution π. The key idea is to recover π as a slice of some Springer resolution. More precisely, let us consider the following two nilpotent orbits in $\mathfrak{sp}_6$:

$$\mathcal{O}_{[2,2,2]} := \mathrm{Sp}_6 \cdot \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{O}_{[4,2]} := \mathrm{Sp}_6 \cdot \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

then their closures in $\mathfrak{sp}_6$ are given by:

$$\overline{\mathcal{O}}_{[2,2,2]} := \{A \in \mathfrak{sp}_6 \mid A^2 = 0\}, \qquad \overline{\mathcal{O}}_{[4,2]} := \{A \in \mathfrak{sp}_6 \mid A^4 = 0\}.$$

We will prove that the wreath product $\mathrm{Sym}^2(\mathbb{C}^2/\pm 1)$ is in fact isomorphic to the intersection of a transverse slice of the nilpotent orbit $\mathcal{O}_{[2,2,2]}$ with the nilpotent orbit closure $\overline{\mathcal{O}}_{[4,2]}$ in $\mathfrak{sp}_6$. The singular variety $\overline{\mathcal{O}}_{[4,2]}$ admits exactly two symplectic resolutions. By restricting them to the transverse slice, we recover exactly the two symplectic resolutions of $\mathrm{Sym}^2(\mathbb{C}^2/\pm 1)$. Using deformations of Springer resolutions (e.g. [Fu]), we construct a symplectic deformation of π.

It is somewhat surprising that we can recover the wreath product $\mathrm{Sym}^2(\mathbb{C}^2/\pm 1)$ from nilpotent orbits, although the interplay between nilpotent orbits and Hilbert schemes has been noticed in [Man], where a transverse slice to the nilpotent orbit $\mathcal{O}_{[2m-n,n]}(n \leq m)$ in the nilpotent cone of $\mathfrak{sl}_{2m}$ is recovered as an open subset (whose complement is of codimension 1 when $n \geq 2$) of the Hilbert scheme $Hilb^n(A_{2m})$, for some singular surface A_{2m}. Here $\mathcal{O}_{[2m-n,n]}$ consists of nilpotent matrices $A \in \mathfrak{sl}_{2m}$ whose Jordan form has only two blocks, with sizes $2m - n$ and n respectively. It would be very interesting to recover other wreath products as a transverse slice to nilpotent orbits, which would in turn reveal more the mysterious relationships between Hilbert-Chow resolutions and Springer resolutions, although the two objects are studied in usual separately.

1. The transverse slice

Let $\mathfrak{g}$ be a simple Lie algebra and G its adjoint group. For any nilpotent element $x \in \mathfrak{g}$, by the theorem of Jacobson-Morozov, there exists an $\mathfrak{sl}_2$-triplet (x, y, h). Then $S = x + \mathfrak{g}^y$ is a transverse slice to the nilpotent orbit $G \cdot x$ in $\mathfrak{g}$, and the morphism $G \times S \to \mathfrak{g}$ is smooth ([Slo], Section 7.4). Here $\mathfrak{g}^y := \{z \in \mathfrak{g} \mid [z, y] = 0\}$.

From now on, let $\mathfrak{g} = \mathfrak{sp}_6$, and consider the following $\mathfrak{sl}_2$-triplet associated to the nilpotent orbit $\mathcal{O}_{[2,2,2]}$:

$$x_0 = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \quad h_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2}$$

where I is the 3×3 identity matrix. Note that $\mathcal{O}_{[2,2,2]} = \mathrm{Sp}_6 \cdot x_0$.

The transverse slice to the orbit $\mathcal{O}_{[2,2,2]}$ is given by

$$S = x_0 + \mathfrak{g}^{y_0} = \{\begin{pmatrix} Z_1 & I \\ Z_2 & Z_1 \end{pmatrix} \mid Z_1 + Z_1^T = 0, Z_2 = Z_2^T\} \subset \mathfrak{sp}_6.$$

We choose the following parameters for Z_1 and Z_2:

$$Z_1 = \begin{pmatrix} 0 & a_3/2 & -a_2/2 \\ -a_3/2 & 0 & a_1/2 \\ a_2/2 & -a_1/2 & 0 \end{pmatrix}, \qquad Z_2 = \begin{pmatrix} x_1 & y_1 & y_2 \\ y_1 & x_2 & y_3 \\ y_2 & y_3 & x_3 \end{pmatrix}.$$

Note that $\mathcal{O}_{[2,2,2]} \subset \overline{\mathcal{O}}_{[4,2]}$, and the codimension is is 4. Let T be the scheme intersection $S \cap \overline{\mathcal{O}}_{[4,2]}$. The variety $\overline{\mathcal{O}}_{[4,2]}$ is normal and the morphism $G \times T \to \overline{\mathcal{O}}_{[4,2]}$ is smooth. It follows that T is normal. As easily seen, a matrix $A \in S$ is in T if and only if $\mathrm{rk}(A) \leq 4$ and $\mathrm{tr}(A) = \mathrm{tr}(A^2) = \mathrm{tr}(A^3) = \mathrm{tr}(A^4) = 0$.

Notice that $\mathrm{rk}(A) \leq 4$ is equivalent to $\mathrm{rk}(Z_2 - Z_1^2) \leq 1$. The matrix $Z_2 - Z_1^2$ is symmetric, so this is equivalent to the existence of $u = (u_1, u_2, u_3) \in \mathbb{C}^3$ such that $Z_2 - Z_1^2 = u^T u$, from which we can substitute the variables x_i, y_j by u_k. Remark that u and $-u$ give the same Z_2, so we should quotient by the following action of $\mathbb{Z}_2$: $u \mapsto -u$.

Now a direct calculus shows that $\mathrm{tr}(A) = \mathrm{tr}(A^3) = 0$. That $\mathrm{tr}(A^2) = 0$ is equivalent to $\sum_{i=1}^3 u_i^2 = \sum_{i=1}^3 a_i^2$, and $\mathrm{tr}(A^4) = 2\,\mathrm{tr}(Z_1^4) + 2\,\mathrm{tr}(Z_2^2) +$

$12\,\mathrm{tr}(Z_1^2 Z_2) = 0$ is equivalent to $\sum_{i=1}^{3} a_i u_i = 0$, which gives:

$$T = \{(a_1, a_2, a_3, u_1, u_2, u_3) \mid \sum_i u_i^2 = \sum_i a_i^2, \sum_i a_i u_i = 0\}/\mathbb{Z}_2,$$

where the action of $\mathbb{Z}_2$ is given by

$$(a_1, a_2, a_3, u_1, u_2, u_3) \mapsto (a_1, a_2, a_3, -u_1, -u_2, -u_3).$$

Consider the following two nilpotent orbits in $\mathfrak{sp}_6$:

$$\mathcal{O}_{[3,3]} = \{A \in \mathfrak{sp}_6 \mid A^3 = 0, \mathrm{rk}(A) = 4\},$$

$$\mathcal{O}_{[4,1,1]} = \{A \in \mathfrak{sp}_6 \mid A^4 = 0, \mathrm{rk}(A) = 3, A^2 \neq 0\}.$$

Then $\mathcal{O}_{[2,2,2]} = \overline{\mathcal{O}}_{[3,3]} \cap \overline{\mathcal{O}}_{[4,1,1]}$ and $\overline{\mathcal{O}}_{[3,3]} \subset \overline{\mathcal{O}}_{[4,2]} \supset \overline{\mathcal{O}}_{[4,1,1]}$. The relationship of inclusions can be resumed in the following diagram:

$$\begin{tikzcd}
 & \overline{\mathcal{O}}_{[4,2]} & \\
\overline{\mathcal{O}}_{[4,1,1]} \arrow[ur] & & \overline{\mathcal{O}}_{[3,3]} \arrow[ul] \\
 & \overline{\mathcal{O}}_{[2,2,2]} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The intersection of T with the two orbit closures $\overline{\mathcal{O}}_{[3,3]}, \overline{\mathcal{O}}_{[4,1,1]}$ is exactly the singular locus of T, which is defined by the following

$$T \cap \overline{\mathcal{O}}_{[3,3]} = \{\sum_i a_i^2 = 0, u^T u = -4Z_1^2\},$$

$$T \cap \overline{\mathcal{O}}_{[4,1,1]} = \{\sum_i a_i^2 = 0, u_1 = u_2 = u_3 = 0\}.$$

Both are isomorphic to the surface $\mathbb{C}^2/\pm 1$ with an isolated A_1-singularity. The intersection $T \cap \mathcal{O}_{[2,2,2]} = x_0$ is just a point.

2. The wreath product

Now we consider the simplest wreath product $\mathrm{Sym}^2(\mathbb{C}^2/\pm 1) = \mathbb{C}^4/H$, where H is the subgroup of $\mathrm{Sp}(4)$ generated by the following elements:

$$\sigma(x_1, x_2, y_1, y_2) = (y_1, y_2, x_1, x_2), \;\; \tau(x_1, x_2, y_1, y_2) = (-x_1, -x_2, y_1, y_2).$$

To write down equations for this affine normal variety, we put

$$a_1 = -i(x_1^2 + y_1^2 + x_2^2 + y_2^2)/2, a_2 = (x_1^2 + y_1^2 - x_2^2 - y_2^2)/2, a_3 = x_1x_2 + y_1y_2,$$

$$u_1 = x_1y_1 + x_2y_2, u_2 = i(x_1y_1 - x_2y_2), u_3 = i(x_1y_2 + x_2y_1).$$

The functions a_1, a_2, a_3 are H-invariant, and the action of H restricts to a $\mathbb{Z}_2$-action on (u_1, u_2, u_3) given by $(u_1, u_2, u_3) \mapsto (-u_1, -u_2, -u_3)$. Now it is straight-ward to check that

$$\mathbb{C}^4/H \simeq \{(a_1, a_2, a_3, u_1, u_2, u_3) \mid \sum a_i^2 = \sum u_i^2,\ \sum_i a_iu_i = 0\}/\mathbb{Z}_2,$$

where the $\mathbb{Z}_2$-action is given by $(a_1, a_2, a_3, u_1, u_2, u_3) \mapsto (a_1, a_2, a_3, -u_1, -u_2, -u_3)$. This gives the following proposition.

Proposition 1. *The transverse slice T is isomorphic to the wreath product $X := \mathrm{Sym}^2(\mathbb{C}^2/\pm 1)$.*

The singular locus of the wreath product X has two components: one is the diagonal Δ and the other will be denoted by Ξ. One sees that the isomorphism between T and X sends $T \cap \overline{\mathcal{O}}_{[3,3]}$ to Δ and $T \cap \overline{\mathcal{O}}_{[4,1,1]}$ to Ξ.

A symplectic resolution of X is given by the composition:

$$\pi : \mathrm{Hilb}^2(T^*\mathbb{P}^1) \to \mathrm{Sym}^2(T^*\mathbb{P}^1) \to \mathrm{Sym}^2(\mathbb{C}^2/\pm 1) = X.$$

The central fiber of π contains a $\mathbb{P}^2$, so we can blow up this $\mathbb{P}^2$ and then blow down along the other direction, i.e. we can perform a Mukai flop, which gives another symplectic resolution $\pi^+ : \mathrm{Hilb}^2(T^*\mathbb{P}^1) \to X$. One sees that $\pi^{-1}(0) \subset \overline{\pi^{-1}(\Delta - \{0\})}$, but $\pi^{-1}(0)$ is not contained in $\overline{\pi^{-1}(\Xi - \{0\})}$, so Δ and Ξ are not symmetric with respect to π. For π^+, it changes the role of Δ and Ξ. It is known that any projective symplectic resolutions of X is isomorphic to π or π^+ (for details, see ([FN], [Fuj])).

3. Springer resolutions

The nilpotent orbit closure $\overline{\mathcal{O}}_{[4,2]}$ admits exactly two symplectic resolutions, given by Springer maps:

$$T^*(G/P_1) \xrightarrow{\phi_1} \overline{\mathcal{O}}_{[4,2]} \xleftarrow{\phi_2} T^*(G/P_2), \tag{3}$$

where P_1 (resp. P_2) is the standard parabolic sub-group of G with flag type [1,2,2,1] (resp. [2,1,1,2]). The matrix forms of P_1 and P_2 are as follows:

$$P_1 = \{ \begin{pmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{pmatrix} \}, \quad P_2 = \{ \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix} \}.$$

The restrictions of ϕ_1, ϕ_2 to the pre-image of the transverse slice T give two projective symplectic resolutions of T.

$$\mathrm{Hilb}^2(T^*\mathbb{P}^1) \simeq Z_1 \xrightarrow{\pi_1} T \xleftarrow{\pi_2} Z_2 \simeq \mathrm{Hilb}^2(T^*\mathbb{P}^1).$$

Proposition 2. *The two symplectic resolutions π_1, π_2 are related by a Mukai flop, in particular, they are not isomorphic. Furthermore $\pi_1 = \pi$ and $\pi_2 = \pi^+$.*

Proof. We will calculate the central fiber over the point $x_0 = T \cap \mathcal{O}_{[2,2,2]}$ (c.f. (2)) under the maps π_1 and π_2. Let $\{e_i, 1 \leq i \leq 6\}$ be the natural basis of $\mathbb{C}^6$ and the symplectic form is $\omega = \sum_{i=1}^3 e_i^* \wedge e_{i+3}^*$. Notice that $Im(x_0) = Ker(x_0) = \mathbb{C}\langle e_1, e_2, e_3\rangle =: K$ is Lagrangian.

It is easy to see that

$$\pi_1^{-1}(x_0) = \{\text{flags } (F_1 \subset F_2) \mid x_0 F_2 \subset F_1 \subset K, F_2 = F_2^{\perp}, \dim F_1 = 1\}.$$

Since $x_0 F_2 \subset F_1$ is of dimension 1, one has two possibilities:

(i). $dim(K \cap F_2) = 2$, then $F_1 = x_0 F_2 \subset x_0 F_1^{\perp}$. Suppose that F_1 is generated by $\sum_{i=1}^3 a_i e_i$, then $x_0 F_1^{\perp} = \{\sum_{i=1}^3 b_i e_i | \sum_i a_i b_i = 0\}$. The condition $F_1 \subset x_0 F_1^{\perp}$ is equivalent to $\sum_{i=1}^3 a_i^2 = 0$, which is a $\mathbb{P}^1$ inside $\mathbb{P}(K)$. The condition for F_2 is just $x_0 F_1^{\perp} \subset F_2 \subset x_0^{-1} F_1$ which is a $\mathbb{P}^1$. So finally this component is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

(ii). $F_2 = K$, then this is isomorphic to $\mathbb{P}(K)$. The two components intersect at a curve $C_1 \simeq \mathbb{P}^1$ inside $\mathbb{P}(K)$.

The fiber of x_0 under π_2 consists of flags $(F_1 \subset F_2)$ such that $x_0 F_2 \subset F_1 \subset K, F_2 = F_2^{\perp}$ and $\dim F_1 = 2$. Since $F_1 \subset K \cap F_2$, so $\dim(K \cap F_2) \geq 2$. There are two cases:

(i). $dim(K \cap F_2) = 2$, then $F_1 = K \cap F_2$ and $x_0 F_2 \subset F_1$. This gives that $x_0 F_2 = x_0 F_1^{\perp} \subset F_1$. Suppose F_1 is generated by $\sum_{i=1}^3 a_i e_i, \sum_{i=1}^3 b_i e_i$. Then we have $x_0 F_1^{\perp} = \{\sum_i c_i e_i | \sum_i a_i c_i = \sum_i b_i c_i = 0\}$. The condition $x_0 F_1^{\perp} \subset F_1$ is equivalent to the existence of $(y, y') \neq (0,0)$ such that $y(\sum_i a_i^2) + y'(\sum_i a_i b_i) = 0$ and $y(\sum_i a_i b_i) + y'(\sum_i b_i^2) = 0$. So the condition for F_1 is $(\sum_i a_i^2)(\sum_i b_i^2) = (\sum_i a_i b_i)^2$. Under the Plücker embedding $\mathbb{P}(\wedge^2 F_1) \to \mathbb{P}(\wedge^2 K) \simeq \mathbb{P}(K^*)$, one sees that this is a conic in $\mathbb{P}^2$. The condition for F_2 turns to be $F_1 \subset F_2 \subset F_1^{\perp}$. So this component is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

(ii). $K = F_2$, then $F_1 \subset K$, this component is just $\mathbb{P}(K^*)$. The two components intersects at a $C_2 \simeq \mathbb{P}^1$ inside $\mathbb{P}(K^*)$.

Now it is clear that the two resolutions are different and are related by the Mukai flop along the component $\mathbb{P}(K^*)$, and C_1, C_2 are dual conics.

Now we will identify π_1, π_2 with π, π^+. By definition, we have

$$\pi_1^{-1}(T \cap \overline{\mathcal{O}}_{[3,3]}) = \{(F_1 \subset F_2, z) \mid zF_2 \subset F_1 \subset Ker(z), F_2 = F_2^{\perp}, \dim F_1 = 1\},$$

where z is in $T \cap \overline{\mathcal{O}}_{[3,3]}$. Consider the elements $z_t \in T \cap \overline{\mathcal{O}}_{[3,3]}$ $(t \in \mathbb{C})$ given by

$$z_t = \begin{pmatrix} tB & I \\ -3t^2 B^2 & tB \end{pmatrix}, \text{with} \quad B = \begin{pmatrix} 0 & \sqrt{-1} & 1 \\ -\sqrt{-1} & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

One has $Ker(z_t) = \mathbb{C}\langle e_1 + \sqrt{-1} t e_5 + t e_6, e_2 - \sqrt{-1} e_3 \rangle$. When t goes to 0, $Ker(z_t)$ goes to $\mathbb{C}\langle e_1, e_2 - \sqrt{-1} e_3\rangle$, thus the limit of $\pi_1^{-1}(z_t)$ will be $\mathbb{P}(\mathbb{C}\langle e_1, e_2 - \sqrt{-1} e_3 \rangle) \subset \mathbb{P}(K)$, which is not a point. This shows that $\pi_1 = \pi$ by the description of π and π^+ in section 2. $\square$

4. Symplectic deformations

A deformation of the symplectic resolutions $\phi_i, i = 1, 2$ (cf. (3)) can be constructed as follows([Fu]). Let $\mathfrak{c}_i$ be the center of the Levi sub-algebra of $\mathfrak{p}_i := Lie(P_i)$ and $\mathfrak{u}_i$ the nil-radical of $\mathfrak{p}_i$. The vector space $V_i := \mathfrak{c}_i + \mathfrak{u}_i$ is a flat family over $\mathfrak{c}_i$. Let $\mathcal{Y}_i$ be the closed sub-variety

$$\mathcal{Y}_i := \{(z, v) \in \mathfrak{c}_i \times G \cdot V_i \mid v \in G \cdot (z + \mathfrak{u}_i)\}.$$

Now consider the morphism $\Phi_i : G \times^{P_i} V_i \to \mathcal{Y}_i$ given by

$$g * (z + u) \mapsto (z, g \cdot (z + u)),$$

where $g \in G, z \in \mathfrak{c}_i$ and $u \in \mathfrak{u}_i$. Notice that if $z \neq 0$, then $z + \mathfrak{u}_i = P_i \cdot z$, so this morphism is well-defined. One can show that Φ_i is birational and it gives a family of morphisms over $\mathfrak{c}_i$. When $z \in \mathfrak{c}_i$ is generic, in the sense that the stabilizer G^z of z is exactly the Levi sub-group L_i of P_i, then

$$\Phi_i^z : G \times^{P_i} (z + \mathfrak{u}_i) \simeq G \times^{P_i} (P_i \cdot z) \to \mathcal{Y}_i^z = G \cdot z \simeq G/L_i$$

is an isomorphism. Notice that $\mathcal{Y}_i^z$ is a semi-simple orbit, thus it is symplectic. When $z = 0$, the map Φ_i^0 is just the Springer resolution ϕ_i. In other words, Φ_i is a symplectic deformation of ϕ_i with base $\mathfrak{c}_i$.

Let $\mathcal{T}_i$ be the intersection $(\mathfrak{c}_i \times S) \cap \mathcal{Y}_i$ and $\mathcal{Z}_i$ its pre-image under the morphism Φ_i, which gives a map $\mathcal{Z}_i \xrightarrow{\Pi_i} \mathcal{T}_i$ over $\mathfrak{c}_i$. Now we will show that the family $\psi_i : \mathcal{Z}_i \to \mathfrak{c}_i$ is smooth. Recall that the map $G \times S \to \mathfrak{g}$ is smooth, so is $G \times (S \cap (G \cdot V_i)) \to G \cdot V_i$ ([Slo] Section 5.1). The morphism Φ_i is G-equivariant, so $G \times \mathcal{Z}_i \to G \times^{P_i} V_i$ is smooth. Notice that the map $G \times^{P_i} V_i \to \mathfrak{c}_i$ is smooth, so is the composition $G \times \mathcal{Z}_i \to \mathfrak{c}_i$. The projection $G \times \mathcal{Z}_i \to \mathcal{Z}_i$ is smooth, which implies the smoothness of $\mathcal{Z}_i \to \mathfrak{c}_i$.

An immediately corollary is that $\mathcal{Z}_i$ is smooth and for any $0 \neq z \in \mathfrak{c}_i$ generic, the intersection $S \cap G \cdot z$ is smooth and symplectic, which deforms $\mathrm{Sym}^2(\mathbb{C}^2/\pm 1)$. So $\mathcal{Z}_i \xrightarrow{\Pi_i} \mathcal{T}_i$ gives a symplectic deformation of the symplectic resolution $\pi_i : \mathrm{Hilb}^2(T^*\mathbb{P}^1) \to \mathrm{Sym}^2(\mathbb{C}^2/\pm 1)$.

5. Universal Poisson deformations

Now we will show that our picture is similar to that of Brieskorn (see section 3 [GK]). We will only consider ϕ_1. To simplify the notations, we will write P (resp. ϕ, L etc.) instead of P_1 (resp. ϕ_1, L_1 etc.).

Fix a maximal torus U in G and a Cartan sub-algebra $\mathfrak{h}$ in $\mathfrak{g} = \mathfrak{sp}_6$. Coordinates in $\mathfrak{h}$ are denoted by (h_1, h_2, h_3). We define the Weyl group of L to be $W(L) := N_L(U)/U$, where $N_L(U)$ is the normalizer of U in L. The partial Weyl group of P is $W^P := N_G(L)/L$. Then W^P is naturally isomorphic to the quotient $N_{W(G)}(W(L))/W(L)$, where $W(G)$ is the Weyl group of G.

It is easy to see that $W(L)$ is isomorphic to $\mathbb{Z}_2$, acting on $\mathfrak{h}$ by $(h_1, h_2, h_3) \mapsto (h_1, h_3, h_2)$. The center $\mathfrak{c}$ of $Lie(L)$ is naturally identified with the fixed point set $\mathfrak{h}^{W(L)}$. The group W^P is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, which acts on $\mathfrak{h}^{W(L)}$ by $(h_1, h_2, h_2) \mapsto (-h_1, h_2, h_2)$ and $(h_1, h_2, h_2) \mapsto (h_1, -h_2, -h_2)$, i.e. it is the sum of two copies of the sign representation of $\mathbb{Z}_2$.

Let $\mathcal{T}'$ be the intersection $S \cap G \cdot (\mathfrak{c} + \mathfrak{u})$, then we have a natural projection

$p : \mathcal{T} \to \mathcal{T}'$ and the following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{Z} & \xrightarrow{\Pi} & \mathcal{T} & \xrightarrow{p} & \mathcal{T}' \\ \psi \downarrow & & \psi' \downarrow & & \beta \downarrow \\ \mathfrak{h}^{W(L)} & \xrightarrow{id} & \mathfrak{h}^{W(L)} & \xrightarrow{\eta} & \mathfrak{h}^{W(L)}/W^P, \end{array} \tag{4}$$

where η is the natural quotient map and β is the restriction to $\mathcal{T}'$ of the adjoint quotient map $\mathfrak{g} \to \mathfrak{h}/W(G)$.

Claim: The second Poisson cohomoloy $HP^2(T)$ can be naturally identified with $\mathfrak{h}^{W(L)}/W^P$.

Let H' be the semi-direct product of $\mathbb{Z}_2$ with W^P. Let $\mathbb{Z}_2$ acts on $\mathfrak{h}^{W(L)}$ by $(h_1, h_2, h_2) \mapsto (h_2, h_1, h_1)$, then it is easy to see that $(\mathfrak{h}^{W(L)} \oplus (\mathfrak{h}^{W(L)})^*)/H'$ is isomorphic to $T \simeq \mathrm{Sym}^2(\mathbb{C}^2/\pm 1)$. By [GK] (section 4), we have $HP^2(T)$ is naturally isomorphic to $HP^2(T \cap \overline{\mathcal{O}}_{[3,3]}) \oplus HP^2(T \cap \overline{\mathcal{O}}_{[4,1,1]})$. By Lemma 3.1 [GK], $HP^2(T \cap \overline{\mathcal{O}}_{[3,3]})$ is naturally identified with $\mathbb{C}v_1/\mathbb{Z}_2$ and $HP^2(T \cap \overline{\mathcal{O}}_{[4,1,1]})$ is identified with $\mathbb{C}v_2/\mathbb{Z}_2$, where $v_1 = (0,1,1), v_2 = (1,0,0)$ are two points in $\mathfrak{h}^{W(L)}$, and the group $\mathbb{Z}_2$ acts by sign representation.

Note that under this identification, $\mathfrak{h}^{W(L)}$ is identified with $H^2(\mathrm{Hilb}^2(T^*\mathbb{P}^1))$. The first square in (4) is the symplectic deformation of π, the second square is Cartesian. For the vertical morphisms, ψ is a universal Poisson deformation of $\mathrm{Hilb}^2(T^*\mathbb{P}^1)$, ψ' is similar to the Calogero-Moser deformation of $T \simeq (\mathfrak{h}^{W(L)} \oplus (\mathfrak{h}^{W(L)})^*)/H'$, and β is the universal Poisson deformation of T.

Remark 1. *A diagram analogous to* (4) *can be constructed using the same method for any Springer resolution.*

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